# Diagonal Matrices, Upper and Lower Triangular Matrices

## Linear Algebra

### MATH 2010

- **Diagonal Matrices:**
  - **Definition:** A *diagonal matrix* is a square matrix with zero entries except possibly on the main diagonal (extends from the upper left corner to the lower right corner).
  - **Examples:** The following are examples, of diagonal matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

  - In general, a diagonal matrix is given by

$$A = \begin{bmatrix} d_1 & 0 & \dots & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \ddots & 0 \\ 0 & \dots & \dots & \dots & d_k \end{bmatrix}$$

  - **Notation:** A lot of the time, a diagonal matrix is referenced with a capital $D$ (for diagonal).
  - **Powers:** If $D$ is a diagonal matrix, then $D^n$ for $n > 0$ is given by

$$D^n = \begin{bmatrix} d_1^n & 0 & \dots & \dots & 0 \\ 0 & d_2^n & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \ddots & 0 \\ 0 & \dots & \dots & \dots & d_k^n \end{bmatrix}$$

  - **Inverses:** A diagonal matrix $D$ is invertible if and only if all the diagonal elements are nonzero. In this case, $D^{-1}$ is given by

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{d_2} & 0 & \dots & 0 \\ 0 & \dots & \ddots & \dots & 0 \\ 0 & \dots & \dots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \frac{1}{d_k} \end{bmatrix}$$

  - **Example:** Let

$$D = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

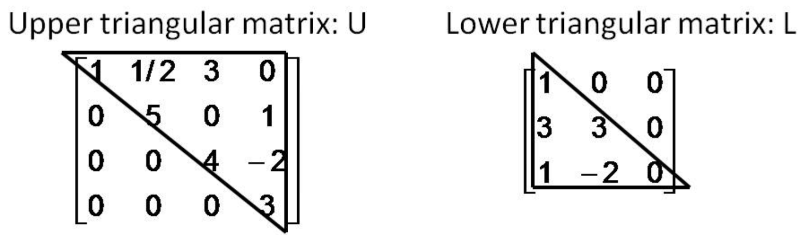
    Then

$$D^3 = \begin{bmatrix} (\frac{1}{2})^3 & 0 & 0 & 0 \\ 0 & 3^3 & 0 & 0 \\ 0 & 0 & 5^3 & 0 \\ 0 & 0 & 0 & (-1)^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 \\ 0 & 0 & 125 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

    and

$$D^{-1} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- **Upper and Lower Triangular Matrices:**
  - **Definition:** An *upper triangular matrix* is a square matrix in which all entries below the main diagonal are zero (only nonzero entries are found above the main diagonal - in the upper triangle). A *lower triangular matrix* is a square matrix in which all entries above the main diagonal are zero (only nonzero entries are found below the main diagonal - in the lower triangle). See the picture below.




  - **Notation:** An upper triangular matrix is typically denoted with $U$ and a lower triangular matrix is typically denoted with $L$.
  - **Properties:**
    1. $\begin{cases} U^T = L \\ L^T = U \end{cases}$

       If you transpose an upper (lower) triangular matrix, you get a lower (upper) triangular matrix.
    2. $\begin{cases} L_1 L_2 = L \\ U_1 U_2 = U \end{cases}$

       The product of two lower (upper) triangular matrices if lower (upper) triangular.
    3. A triangular matrix is invertible if and only if all diagonal entries are nonzero.

$$\begin{bmatrix} 1 & -5 & 3 & 4 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ is NOT invertible, and } \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \text{ IS invertible.}$$